ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 45–71

BIINVERTIBLE ACTIONS OF HOPF ALGEBRAS

BY

S. MONTGOMERY*

Department of Mathematics
University of Southern California
Los Angeles, CA 90089-1113, USA

ABSTRACT

We study actions of a Hopf algebra H on an algebra R such that the action is twisted by an invertible map σ: $H \otimes H \to R$; the biinvertible condition means that these actions also have both an inverse and an antiinverse in $\mathrm{Hom}(H, \mathrm{End}\, R)$. When R is an ordinary H-module algebra, the action is biinvertible if the antipose is bijective. As a new example we show that if the H-action is twisted and the coradical of H is cocommutative, then the action is biinvertible. After studying the continuity of these actions with respect to the filter of ideals of R with zero annihilator, we consider when the actions may be extended to the symmetric Martindale quotient ring of R and its H-analog. Our results can be applied to crossed products $R\#_\sigma H$.

Introduction

Usually in studying an action of a Hopf algebra H on an algebra R it is assumed that R is an H-module. In constructing a crossed product $R\#_\sigma H$ of H over R, however, this is not necessarily true, since the action is twisted by the cocycle σ; this already happens in the case of crossed products of groups acting on non-commutative rings. In fact when H is cocommutative and $R\#_\sigma H$ is a crossed product, it is shown in [BCM] that R is an H-module if and only if σ has values in the center of R.

In this paper we consider these more general actions which are twisted by a map σ: $H \otimes H \to R$ which is convolution-invertible, although we do not require that σ

* Research supported by NSF Grant No. 89-01491.
Received July 18, 1991 and in revised form June 25, 1992

is a cocycle; in the terminology of [Sc], R is a twisted H-module algebra. We show that many desirable properties of H-module algebras can be recovered provided the H-action on R is invertible and anti-invertible, considered as an element of the convolution algebra Hom(H, End R). We call such actions biinvertible; examples include the usual H-module algebras for any H whose antipode is bijective, and also any twisted action by a Hopf algebra whose coradical is cocommutative. Such Hopf algebras include group algebras, enveloping algebras of Lie algebras, and the regular functions on a unipotent algebraic group. A newer example is given by $U_q(g)$, the quantum enveloping algebra of a complex semi-simple Lie algebra g; in fact these Hopf algebras are pointed, with coradical the group algebra of the obvious group-like elements. This is shown in [M2]; see also [R, Lemma 1].

Our main results concern when twisted actions may be extended to the symmetric Martindale quotient ring of R and its H-analog. The importance of this property is that it enables one to define X-inner actions, that is actions which become inner on the quotient. For groups such actions, introduced by Kharchenko, proved very useful in sudying fixed rings and crosssed products, since frequently one could reduce to the X-inner and X-outer cases (see [M], [P]). Such X-inner actions for Hopf algebras were studied in [C] for H-module algebras and in [Ch] for crossed products with anti-invertible actions. Other recent work on X-inner H-actions appears in [Sc].

This paper is organized as follows. In Section 1 we formally define a twisted action φ, its inverse θ and anti-inverse ψ, as well as crossed product actions and the continuity of these maps with respect to various filters of ideals of R. We then prove some basic properties of these maps, in particular that θ and ψ are themselves "twisted" actions in a certain sense.

Section 2 is concerned with examples of biinvertible actions, and we prove the result mentioned above: if H has cocommutative coradical H_0, then any twisted action of H is biinvertible. Moreover if H is pointed (that is H_0 is a group algebra), then the maps φ, θ and ψ are all continuous with respect to the filter $\mathcal{F}$ of all ideals of R with zero annihilator.

In Section 3 we show that biinvertible twisted actions can be extended uniquely to the symmetric Martindale quotient ring Q of R, provided θ and ψ are $\mathcal{F}$-continuous (respectively, extended to the H-quotient ring Q_H of R relative to the filter $\mathcal{F}_H = \mathcal{F} \cap \{H\text{-stable ideals}\}$, provided θ and ψ are $\mathcal{F}_H$-continuous). This extends results of [C] and [Ch]; both considered only Q_H, Cohen for H-module algebras and Chin for "fully antiinvertible" crossed product actions. By the results of Section 2, these results apply to all pointed Hopf algebras; moreover, the extension of the action φ to Q or Q_H is also biinvertible in that case. In general, however, we do not know if the extension of φ is biinvertible, since extending θ and ψ is more difficult.

In the last section we specialize to crossed products and give a criterion for invertibility of the action in terms of maps from H to $R\#_\sigma H$; this parallels Chin's criterion for antiinvertibility. We do not know if these criteria are necessary for biinvertibility. Finally when the crossed product action extends to Q_H, we may construct a larger crossed product $Q_H(R)\#_\sigma H$; we show that this new algebra embeds naturally into $Q(R\#_\sigma H)$. This result is used in [BeM], where it is proved that if H is irreducible and every non-zero H-stable ideal of R contains a regular element, then $Q(R\#_\sigma H) = Q_H(R)\#_\sigma H$.

Throughout H denotes a Hopf algebra over the field k, and R a k-algebra. We will follow the notation in Sweedler's book [S]; thus H has comultiplication $\Delta\colon H \to H \otimes H$, counit $\varepsilon\colon H \to K$, and antipode $S\colon H \to H$. The unit element in H may be written as a map $\eta\colon k \to H$. We usually abbreviate the summation notation $\Delta h = \sum_h h_{(1)} \otimes h_{(2)}$ by the notation $\Delta h = \sum_h h_1 \otimes h_2$. Also recall that $\Delta_{n+1}\colon H \to H^{\otimes^{n+2}}$ is defined inductively by $\Delta_1 = \Delta$ and $\Delta_{n+1} = (\Delta \otimes I^n) \circ \Delta_n$.

1. Definitions and preliminaries

The notion of biinvertibility makes sense for coalgebras, and not just for Hopf algebras. Thus let C be a k-coalgebra and R a k-algebra. Recall that C **measures** R if there exists a k-linear map $\varphi\colon C \to \mathrm{End}_k(R)$, written $\varphi_c a = c \cdot a$, all $c \in C, a \in R$, such that $c\cdot(ab) = \sum_c (c_1 \cdot a)(c_2 \cdot b)$ and $c \cdot 1 = \varepsilon(c)1$, $c \in C, a, b \in R$. We wish to consider φ as an element of the convolution algebra $\mathrm{Hom}_k(C, \mathrm{End}\ R)$.

1.1 Definition: Let φ be a measuring of R by C.

(1) φ is **invertible** if it has an inverse $\theta \in \mathrm{Hom}(C, \mathrm{End}\ R)$. That is,

$$\sum_c \theta_{c_1} \circ \varphi_{c_2} = \varepsilon(c) id_R = \sum_c \varphi_{c_1} \circ \theta_{c_2}, \quad \text{all } c \in C.$$

(2) φ is **anti-invertible** if it has an anti-inverse $\psi \in \mathrm{Hom}(C, \mathrm{End}\ R)$. That is,

$$\sum_c \psi_{c_2} \circ \varphi_{c_1} = \varepsilon(c) id_R = \sum_c \varphi_{c_2} \circ \psi_{c_1}, \quad \text{all } c \in C.$$

(3) φ is **biinvertible** if it is both invertible and anti-invertible.

We now assume $C = H$ is a Hopf algebra. The next definition weakens the notion of H-module algebras.

1.2 Definition: Let φ be a measuring of R by H.

(1) φ is called a **twisted action** if

(i) $1 \cdot a = a$, all $a \in R$, and

(ii) there exists an invertible map $\sigma \in \mathrm{Hom}_k(H \otimes H, R)$ such that

$$(1.3) \qquad h \cdot (k \cdot a) = \sum_{h,k} \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3)$$

all $h, k \in H$, $\quad a \in R$.

(2) φ is called a **crossed product action** if it is a twisted action and in addition

(iii) $\sigma(h,1) = \sigma(1,h) = \varepsilon(h)1$, all $h \in H$,

(iv) σ satisfies the cocycle condition

$$(1.4) \qquad \sum_{h,k,m} [h_1 \cdot \sigma(k_1, m_1)]\sigma(h_2, k_2 m_2) = \sum_{h,k} \sigma(h_1, k_1)\sigma(h_2 k_2, m)$$

all $h, k, m \in H$.

(1.3) is called the **twisted module condition** (TMC), and if φ is a twisted action of H on R we also say that R is a **twisted H-module algebra**, as in [Sc]. A crossed product action is precisely what is required to form the crossed product algebra $R\#_\sigma H$ [DT, BCM]; as a vector space, this algebra is isomorphic to $R \otimes H$, and it has multiplication given by

$$(1.5) \qquad (a\#h)(b\#k) = \sum_{h,k} a(h_1 \cdot b)\sigma(h_2, k_1)\#h_3 k_2,$$

all $h, k \in H$, $\quad a, b \in R$, with identity element $1\#1$.

Now let $\mathcal{F}$ denote the filter of all (two-sided) ideals of R with zero left and right annihilator, and let $\mathcal{F}_H = \{I \in \mathcal{F} \mid H \cdot I \subset I\}$, the subfilter of H-stable ideals in $\mathcal{F}$.

1.6 Definition: Let $\Gamma \in \mathrm{Hom}(H, \mathrm{End}\ R)$. Then Γ is **$\mathcal{F}$-continuous** (respectively **$\mathcal{F}_H$ -continuous**) if given any $I \in \mathcal{F}$ (resp. $I \in \mathcal{F}_H$) and $h \in H$, there exists $J \in \mathcal{F}$ (resp. $J \in \mathcal{F}_H$) such that $\Gamma_h J \subseteq I$.

Note that Γ being $\mathcal{F}$-continuous is equivalent to saying that given I, the inverse image of Γ_h on I contains some $J \in \mathcal{F}$; this is just the usual definition of continuity in a topological group, where R is a group under addition and $\mathcal{F}$ is a basis of neighborhoods of 0 in R. Of course these remarks apply to $\mathcal{F}_H$, as does the next lemma.

1.7 LEMMA:

(1) *If Γ is $\mathcal{F}$-continuous, then given $I \in \mathcal{F}$ and $h_1, \dots, h_n \in H$, there exists $J \in \mathcal{F}$ such that $\Gamma_{h_i} J \subseteq I$, for $i = 1, \dots, n$.*

(2) *The $\mathcal{F}$-continuous maps in* $\mathrm{Hom}_k(C, \mathrm{End}\ R)$ *form a subalgebra.*

Proof: (1) is clear since $\mathcal{F}$ is closed under finite intersections. (2) The sum of continuous maps is continuous, as in (1). For the (convolution) product, choose $f, g \in \mathrm{Hom}(H, \mathrm{End}\, R)$ which are $\mathcal{F}$-continuous, $I \in \mathcal{F}$, and $h \in H$. Write

$$\Delta h = \sum_i h_i \otimes h_i' \in H \otimes H.$$

Then

$$(f * g)_h(r) = \sum_i f_{h_i} \circ g_{h_i'}(r), \quad \text{all } r \in R.$$

By (1) we may find $J \in \mathcal{F}$ such that $f_{h_i} J \subseteq I$, all h_i and then find $K \in \mathcal{F}$ such that $g_{h_i'} K \subseteq J$, all h_i'. Thus for $a \in K$,

$$(f * g)_h(a) = \sum_i f_{h_i}(g_{h_i'} a) \subseteq \sum_i f_{h_i} J \subseteq I. \qquad \blacksquare$$

We also consider a property stronger than $\mathcal{F}_H$-continuity.

1.8 Definition: A map $\Gamma \in \mathrm{Hom}(H, \mathrm{End}\, R)$ is **fully H-stabilizing** if $\Gamma_h I \subseteq I$ for all H-stable ideals I of R and all $h \in H$. An action φ of H on R is **fully biinvertible** if it is biinvertible and both θ and ψ are fully H-stabilizing.

The terminology follows Chin [Ch], who studied crossed product actions which were fully anti-invertible (although he called them fully "invertible", only ψ was considered).

1.9 Example: Let R be an H-module algebra and assume the antipode S of H is bijective. Then the action islways fully biinivertible: let $\theta_h a = Sh \cdot a$ and $\psi_h a = S^{-1}h \cdot a$. Thus biinvertibility is a natural generalization of the property of R being an H-module.

1.10 Example: Let $H = kG$, the group algebra of the group G. A measuring of R by H corresponds to a map $\varphi \in \mathrm{Hom}_k(H, \mathrm{End}\, R)$ such that $\varphi(G) \subseteq Alg(R)$, the algebra maps of R to itself. Write $\varphi(g) = \bar{g}$, for $g \in G$. If the measuring is twisted action, then $\varphi(G) \subseteq \mathrm{Aut}(R)$ (see 2.2) and by (1.3) $\bar{g}\bar{h} \equiv \overline{gh}$ (mod $\mathrm{Inn}\, R$), where $\mathrm{Inn}\, R$ denotes the inner automorphisms of R. Conversely, any group homomorphism $f\colon G \to \mathrm{Aut}\, R/\mathrm{Inn}\, R$ gives rise to a twisted action of kG on R as follows: let $\bar{g} \in \mathrm{Aut}\, R$ be a coset representative for $f(g)$; then $\bar{g}\bar{h} = n(g,h)\overline{gh}$ where $n(g,h) \in \mathrm{Inn}\, R$. Let $\sigma(g,h) \in R$ be a unit inducing $n(g,h)$. Then $\varphi(g) = \bar{g}$ is a twisted action of G on R. Note that this construction enables us to give examples of twisted actions which are not crossed product actions.

We remark that any twisted kG-action is $\mathcal{F}$-continuous. For, given $I \in \mathcal{F}$ and $x \in G$, let $J = x^{-1} \cdot I$. Using (1.3),

$$x \cdot (x^{-1} \cdot I) = \sigma(x, x^{-1}) \ (xx^{-1} \cdot I)\sigma^{-1}(x, x^{-1}) \subseteq I.$$

For $h = \Sigma \alpha_x x \in kG$, use $J = \bigcap_x x^{-1} \cdot I$ as in 1.7 (1).

We record some elementary properties of the maps θ and ψ.

1.11 Lemma: *Let φ be a measuring of R by the coalgebra C and let θ and ψ be as in Definition 1.1. Then for all $a, b \in R, h \in C$,*

(1) *θ anti-measures R; that is $\theta_h(ab) = \sum_h (\theta_{h_2} a) \ (\theta_{h_1} b)$,*
(2) *ψ anti-measures R,*
(3) $(h \cdot a)b = \sum_h h_1 \cdot [a(\theta_{h_2} b)]$,
(4) $b(h \cdot a) = \sum_h h_2 \cdot [(\psi_{h_1} b)a]$,
(5) $b(\theta_h a) = \sum \theta_{h_1}[(h_2 \cdot b)a]$,
(6) $(\psi_h a)b = \sum \psi_{h_2}[a(h_1 \cdot b)]$.

Proof:

(1) $\theta_h(ab) = \sum_h \theta_{h_1}[(\varphi_{h_2}\theta_{h_3} a)b] = \sum \theta_{h_1}[(\varphi_{h_2}\theta_{h_5} a) \ (\varphi_{h_3}\theta_{h_4} a)]$
$= \sum \theta_{h_1}\varphi_{h_2}[(\theta_{h_4} a)(\theta_{h_3} b)] = \sum (\theta_{h_2} a)(\theta_{h_1} b)$.

(2) is [C, 1.2]. Parts (3) and (4) are immediate from the definitions, and (5) and (6) follow from (1) and (2) respectively. ∎

1.12 Corollary: *Let φ be a biinvertible measuring of R by H, and let I be an ideal of R.*

(1) *If φ is a twisted action, then $H \cdot I$ is an H-stable ideal of R containing I.*

(2) *If φ is fully biinvertible and I is H-stable, then the right and left annihilators of I are also H-stable. Thus if R is H-prime, $\mathcal{F}_H$ is the set of all nonzero H-stable ideals.*

Proof: (1) That $H \cdot I$ is a right ideal follows from 1.11 (3) and that it is a left ideal follows from 1.11 (4). Thus $H \cdot I$ is an ideal; it contains I since $1 \cdot a = a$, all $a \in R$. To see it is H-stable, we use (1.3): for $h, k \in H$ and $a \in I$,

$$h \cdot (k \cdot a) = \sum \sigma(h_1, k_1)(h_2 k_2 \cdot a)\sigma^{-1}(h_3, k_3) \in R(H \cdot I)R \subseteq H \cdot I.$$

(2) That the right annihilator is H-stable is shown in [Ch, 2.2], using ψ. A similar argument works on the left using θ: let B be the left annihilator of I. Then for $a \in I, b \in B, h \in H$, 1.11 (3) gives $(h \cdot b)a = \sum_h h_1 \cdot [b(\theta_{h_2} a)] \in H \cdot bI = H \cdot 0 = 0$. Here we have used that $\theta_H I \subseteq I$, since φ is fully biinvertible. Finally, since R being H-prime means that the product of non-zero H-stable ideals is non-zero, the fact about $\mathcal{F}_H$ is clear. ∎

The following useful identities can be considered as versions of the twisted module condition (1.3) for the maps θ and ψ.

1.13 PROPOSITION: *Let φ be a twisted action of H on R. Then for all $h, k \in H$, $a, b \in R$,*

(1) $\sum_{h,k} (\theta_{h_3 k_3} a)\, \theta_{k_1} \theta_{h_1} \sigma(h_2, k_2) = \sum_{h,k} \theta_{k_1} \theta_{h_1} (\sigma(h_2, k_2)\, a)$,

(2) $\sum_{h,k} (\psi_{k_3} \psi_{h_3} \sigma^{-1}(h_2, k_2))(\psi_{h_1 k_1} b) = \sum_{h,k} \psi_{k_2} \psi_{h_2} (b \sigma^{-1}(h_1, k_1))$.

Proof: (1) Condition (1.3) is equivalent to

$$\sum h_1 \cdot (k_1 \cdot c)\, \sigma(h_2, k_2) = \sum \sigma(h_1, k_1)(h_2 k_2 \cdot c), \text{ for all } c \in R.$$

Thus

$$\sum h_{11} \cdot (k_{11} \cdot (\theta_{h_2 k_2} a))\sigma(h_{12}, k_{12}) = \sum \sigma(h_{11}, k_{11}) h_{12} k_{12} \cdot (\theta_{h_2 k_2} a),$$

so

$$\sum h_1 \cdot (k_1 \cdot (\theta_{h_3 k_3} a))\sigma(h_2, k_2) = \sum \sigma(h_1, k_1) h_2 k_2 \cdot (\theta_{h_3 k_3} a) = \sigma(h, k)a.$$

Then

$$\sum_{h,k} \theta_{k_1} \theta_{h_1} [h_2 \cdot (k_2 \cdot (\theta_{h_4 k_4} a))\sigma(h_3, k_3)] = \sum_{h,k} \theta_{k_1} \theta_{h_1} (\sigma(h_2, k_2) a).$$

Since θ anti-measures R, the left side equals

$$\sum_{h,k} \theta_{k_{12}}\theta_{h_{12}}(h_2 \cdot (k_2 \cdot (\theta_{h_4 k_4} a)))\theta_{k_{11}}\theta_{h_{11}}(\sigma(h_3, k_3)) = \sum_{h,k}(\theta_{h_3 k_3} a)\theta_{k_1}\theta_{h_1}\sigma(h_2, k_2).$$

(2) Here we use that (1.3) is equivalent to

$$\sum \sigma^{-1}(h_1, k_1)h_2 \cdot (k_2 \cdot c) = \sum (h_1 k_1 \cdot c)\sigma^{-1}(h_2, k_2)$$

for any $c \in R$. Thus

$$\sum_{h,k} \sigma^{-1}(h_{21}, k_{21})h_{22} \cdot (k_{22} \cdot (\psi_{h_1 k_1} b)) = \sum_{h,k}(h_{21}k_{21} \cdot (\psi_{h_1 k_1} b))\sigma^{-1}(h_{22}, k_{22})$$

or

$$\sum_{h,k} \sigma^{-1}(h_2, k_2)h_3 \cdot (k_3 \cdot (\psi_{h_1 k_1} b)) = \sum_{h,k}(h_2 k_2 \cdot (\psi_{h_1 k_1} b))\sigma^{-1}(h_3, k_3) = b\sigma^{-1}(h, k).$$

Then

$$\sum_{h,k} \psi_{k_4}\psi_{h_4}[\sigma^{-1}(h_2, k_2)h_3 \cdot (k_3 \cdot (\psi_{h_1 k_1} b))] = \sum_{h,k} \psi_{k_2}\psi_{h_2}(b\sigma^{-1}(h_1, k_1)).$$

Since ψ anti-measures R, the left side equals

$$\sum_{h,k} \psi_{k_{42}}\psi_{h_{42}}\left(\sigma^{-1}(h_2, k_2)\right)\psi_{k_{41}}\psi_{h_{41}}(h_3 \cdot (k_3 \cdot (\psi_{h_1 k_1} b)))$$
$$= \sum [\psi_{k_3}\psi_{h_3}\sigma^{-1}(h_2, k_2)](\psi_{h_1 k_1} b). \qquad \blacksquare$$

2. Biinvertibility, continuity and the coradical

In this section we show that if the coradical of H is cocommutative, then any twisted action φ of H is fully biinvertible; if in addition H is pointed, then φ, θ, and ψ are all $\mathcal{F}$-continuous.

We first recall some elementary facts about coradicals (see [S]). For any coalgebra C, the **coradical** C_0 is the sum of the simple subcoalgebras. C is **pointed** if all simple subcoalgebras are one-dimensional; when $C = H$, a Hopf algebra, this is equivalent to saying that $H_0 = kG$, the group algebra of the set $G = G(H)$ of group-like elements of H. In general C_0 determines the **coradical filtration** $\{C_n\}$ of C; $C_n \subseteq C_{n+1}$, all n, $C = \bigcup_{n \geq 0} C_n$, and

$$\Delta C_n \subseteq \sum_{i=0}^{n} C_i \otimes C_{n-i}.$$

C_n can be described inductively via $C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C)$ (see [S, 9.0.0]).

2.1 LEMMA: *Let φ be a twisted action of H on R, and let C be an S-stable cocommutative subcoalgebra of H. Then φ is fully biinvertible when restricted to C. For any $h \in C$, the inverse θ of φ is given by*

$$\theta_h a = \sum_h Sh_1 \cdot \left[\sigma^{-1}(h_2, Sh_3) a \ \sigma(h_4, Sh_5)\right].$$

Proof: First, using that S is the (convolution) inverse of id_R, it follows from (1.3) that for all $a \in R, h \in C$,

(1) $\sum_h Sh_1 \cdot (h_2 \cdot a) = \sum_h \sigma(Sh_1, h_2) a \ \sigma^{-1}(Sh_3, h_4)$,

(2) $\sum_h h_1 \cdot (Sh_2 \cdot a) = \sum_h \sigma(h_1, Sh_2) a \ \sigma^{-1}(h_3, Sh_4)$.

It now follows from (2) that $(\varphi * \theta)(h)a = a$, all $a \in R, h \in C$. For

$$\begin{aligned}\sum \varphi_{h_1} \circ \theta_{h_2}(a) &= \sum_h h_1 \cdot \left(Sh_2 \cdot \left[\sigma^{-1}(h_3, Sh_4) a \ \sigma(h_5, Sh_6)\right]\right) \\ &= \sum_h \sigma(h_1, Sh_2) \sigma^{-1}(h_3, Sh_4) a \ \sigma(h_5, Sh_6) \sigma^{-1}(h_7, Sh_8) \\ &= a.\end{aligned}$$

We next claim that $\sum_h \sigma^{-1}(Sh_1, h_2)[Sh_3 \cdot \sigma(h_4, Sh_5)]$ is in the center of R. For, using (3) applied to Sh it follows that

$$\begin{aligned}\sum_h Sh_1 \cdot (h_2 \cdot a) &= \sum_h Sh_1 \cdot (h_2 \cdot (Sh_3 \cdot \theta_{Sh_4} a)) \\ &= \sum_h Sh_1 \cdot [\sigma(h_2, Sh_3)(\theta_{Sh_4} a)\sigma^{-1}(h_5, Sh_6)] \\ &= \sum_h [Sh_1 \cdot \sigma(h_2, Sh_3)] a [Sh_4 \cdot \sigma^{-1}(h_5, Sh_6)].\end{aligned}$$

Comparing this with (1) and using that $\sum_h Sh_1 \cdot \sigma^{-1}(h_2, Sh_3)$ has convolution inverse $\sum_h Sh_1 \cdot \sigma(h_2, Sh_3)$ in $\mathrm{Hom}(C \otimes C \otimes C, R)$ proves the claim.

Finally, we check that $(\theta * \varphi)(h)a = a$, all $a \in R, h \in C$.

$$\begin{aligned}&\sum_h \theta_{h_1}(h_2 \cdot a) \\ &= \sum_h Sh_1 \cdot [\sigma^{-1}(h_2, Sh_3)(h_4 \cdot a)\sigma(h_5, Sh_6)] \\ &= \sum_h [Sh_1 \cdot \sigma^{-1}(h_2, Sh_3)] Sh_4 \cdot (h_5 \cdot a)[Sh_6 \cdot \sigma(h_7, Sh_8)] \\ &= \sum_h [Sh_1 \cdot \sigma^{-1}(h_2, Sh_3)] \sigma(Sh_4, h_5) a \ \sigma^{-1}(Sh_6, h_7)[Sh_8 \cdot \sigma(h_9, Sh_{10})] \\ &= a\end{aligned}$$

using the claim. Thus φ is invertible on C. Since C is cocommutative, $\psi = \theta$ on C and so φ is biinvertible. It is clear from the form of θ_h that it preserves any C-stable ideal of R. ∎

2.2 COROLLARY: *Let φ be a twisted action of $H = kG$ on R. Then φ is fully biinvertible and the elements of G act as automorphisms of R.*

Proof: The first statement follows from 2.1, since $H = C$ is cocommutative. Now if $g \in G$, $\theta_g a = g^{-1} \cdot [\sigma^{-1}(g, g^{-1})a \;\; \sigma(g, g^{-1})]$; here $\sigma^{-1}(g, g^{-1}) = \sigma(g, g^{-1})^{-1}$. Now $g \cdot \theta_g a = a$ gives that g is onto, and $\theta_g(g \cdot a) = a$ gives that g is one-to-one. Thus $\bar{g} = \varphi(g)$ is an automorphism of R. ∎

2.3 PROPOSITION: *Let φ be a twisted action of H on R. If H is pointed, then φ is $\mathcal{F}$-continuous.*

Proof: We use the Taft–Wilson theorem which describes the coradical filtration of H [TW]. As noted above, $H_0 = kG$, where $G = G(H)$. For $n \geq 1$, and $h \in H_n$, $h \notin H_{n-1}$, we may write

$$h = \sum_{x,y \in G} h_{x,y}, \quad \text{where } \Delta h_{x,y} \in x \otimes h_{x,y} + h_{x,y} \otimes y + H_{n-1} \otimes H_{n-1}.$$

This means that $\bmod H_{n-1}$, h acts on R as a sum of x, y-derivations. We proceed by induction on n to show that if $h \in H_n$ and $I \in \mathcal{F}$, then there exists $J \in \mathcal{F}$ with $h \cdot J \subseteq I$.

The case $n = 0$ has already been shown in Example 1.10. Assume it is true for $n-1$, and choose $h \in H_n$, $h \notin H_{n-1}$. Since h is a finite sum of $h_{x,y}$, it suffices to assume $h = h_{x,y}$, some $x, y \in G$. Write

$$\Delta h = x \otimes h + h \otimes y + \sum_{i=1}^{m} z_i \otimes w_i, \qquad \text{where } z_i, w_i \in H_{n-1}.$$

Given $I \in \mathcal{F}$, we may choose $J \in \mathcal{F}$ such that $z_i \cdot J \subseteq I$, all $i = 1, \ldots, m$ by induction on n. Let $K = J \cap (x^{-1} \cdot I) \cap (y^{-1} \cdot I) \in \mathcal{F}$, and choose $a, b \in K$. Then

$$\begin{aligned} h \cdot (ab) &= (x \cdot a)(h \cdot b) + (h \cdot a)(y \cdot b) + \sum_i (z_i \cdot a)(w_i \cdot b) \\ &\subseteq I(h \cdot b) + (h \cdot a)I + \sum_i I(w_i \cdot b) \subseteq I. \end{aligned}$$

Thus $h \cdot K^2 \subseteq I$. Since $K^2 \in \mathcal{F}$, the result is proved. ∎

The next lemma is due to Takeuchi; we include the proof since it is the actual construction of the inverse which we shall need.

2.4 LEMMA ([T, Lemma 14]): *For A an algebra and C a coalgebra, a map $f \in \text{Hom}(C, A)$ is invertible (under convolution) if and only if $f|_{C_0}$ is inertible in* $\text{Hom}(C_0, A)$.

Proof: Assume $f|_{C_0}$ is invertible in $\text{Hom}(C_0, A)$, and let $g \in \text{Hom}(C_0, A)$ be its inverse. We may extend g to an element $g' \in \text{Hom}(C, A)$, by defining $g' = 0$ on a vector space complement of C_0. First set $\gamma = \eta \circ \varepsilon - f * g'$; then $\gamma|_{C_0} = 0$. By induction on n it follows that $\gamma^{n+1} = 0$ on C_n, where γ^n is the n^{th} power of γ in $\text{Hom}(C, A)$. Thus $\sum_{n=0}^{\infty} \gamma^n$ is well-defined, and it is an inverse of $f * g'$. Thus f has a right inverse $g' * \sum_{n \geq 0} \gamma^n$. Similarly, setting $\mu = \eta \circ \varepsilon - g' * f$, we see $\sum_{n=0}^{\infty} \mu^n$ is an inverse for $g' * f$, and thus f has a left inverse. Thus f is invertible in $\text{Hom}(C, A)$. ∎

2.5 PROPOSITION: *Let φ be a measuring of R by the coalgebra C, and θ and ψ as in 1.1. Then*

(1) *φ is invertible (resp. anti-invertible) in $\text{Hom}(C, \text{End}\, R) \Leftrightarrow \varphi|_{C_0}$ is invertible (resp. anti-invertible) in $\text{Hom}(C_0, \text{End}\, R)$.*

(2) *θ (resp. ψ) is fully H-stabilizing $\Leftrightarrow \theta|_{C_0}$ (resp. $\psi|_{C_0}$) is fully H-stabilizing.*

(3) *θ (resp. ψ) is $\mathcal{F}_H$-continuous $\Leftrightarrow \theta|_{C_0}$ (resp. $\psi|_{C_0}$) is $\mathcal{F}_H$-continuous.*

(4) *Assume φ is $\mathcal{F}$-continuous. Then θ (resp. ψ) is $\mathcal{F}$-continuous $\Leftrightarrow \theta|_{C_0}$(resp. $\psi|_{C_0}$) is $\mathcal{F}$-continuous.*

Proof: We use 2.4 with $A = \text{End}\, R$. Then the invertible part of (1) follows immediately. For the anti-inverse ψ of φ, replace C by its opposite coalgebra C^{op}; then C^{op} anti-measures R, and φ determines an element φ^{op} in $\text{Hom}(C^{op}, \text{End}\, R)$. Then 2.4 gives an inverse of φ^{op} in $\text{Hom}(C^{op}, \text{End}\, R)$; this inverse becomes an anti-inverse of φ in $\text{Hom}(C, \text{End}\, R)$. This proves (1). We note that the use of C^{op} also applies in (2) – (4), and thus it will suffice to consider θ.

We prove (4), as (2) and (3) are almost identical (although (2) is a little easier). Thus assume $\theta|_{C_0}$ is $\mathcal{F}$-continuous. Using $f = \varphi$ and $g|_{C_0}$ in 2.4, note that g' is also $\mathcal{F}$-continuous on C. Since f and g' are $\mathcal{F}$-continuous, it follows that $f * g'$ and $g' * f$ are $\mathcal{F}$-continuous by 1.7(2), and thus that γ and μ are $\mathcal{F}$-continuous. By induction all γ^n are $\mathcal{F}$-continuous, and since $\gamma^{n+1} = 0$ on C_n, $\sum_{n \geq 0} \gamma^n$ is $\mathcal{F}$-continuous. Thus the map $g' * \sum_{n \geq 0} \gamma^n$ is $\mathcal{F}$-continuous; this map is θ by uniqueness of inverses. ∎

2.6 THEOREM: *Let φ be a twisted action of H on R.*

(1) *If H_0 is cocommutative, then φ is fully biinvertible,*

(2) *If H is pointed, then also φ, θ and ψ are $\mathcal{F}$-continuous.*

Proof: 2.1, 2.3 and 2.5, since $\theta = \psi$ on H_0. ∎

Thus all our work here applies to the known pointed Hopf algebras; in particular this includes any Hopf algebra of the form $K\#kG$, where K is irreducible, as well as the examples mentioned in the introduction.

3. Extending twisted actions to quotient rings

We consider here several quotient rings of the ring R. First, we use the filter $\mathcal{F}$ of ideals of R which have zero annihilator, as in Section 1.

Let Q^ℓ (respectively Q^r, Q) be the left (respectively right, symmetric) Martindale quotient ring with respect to $\mathcal{F}$, see [P, Chapter 3] for the case of prime rings; arbitrary rings are considered in [A]. R embeds into Q^ℓ(resp. Q^r) as right (left) multiplications, and any $q \in Q^\ell$ (resp. Q^r) has the property that there exists $I \in \mathcal{F}$ such that $Iq \subseteq R$ (resp. $qI \subseteq R$). The symmetric quotient ring can then be described as

$$(3.1)\qquad \begin{aligned} Q &= \{q \in Q^\ell | qI \subseteq R, \text{ some } I \in \mathcal{F}\} \\ &= \{q \in Q^r | Iq \subseteq R, \text{ some } I \in \mathcal{F}\}. \end{aligned}$$

As in [P], it is also seen to be

$$(3.2)\qquad Q = \varinjlim_{I \in \mathcal{F}} \{(f,g) \mid f\colon {}_RI \to R,\ g: I_R \to R \text{ and } (af)b = a(gb), \quad \text{all } a,b \in I, \text{ for some } I \in \mathcal{F}\}.$$

Here g is written on the left and f on the right. In this formulation $R \hookrightarrow Q$ via $a \mapsto (r_a, \ell_a)$, where r_a (resp. ℓ_a) denotes right (left) multiplication by a.

When H acts on R, a smaller quotient ring may be constructed, which will be very useful in studying crossed products. Thus we repeat the above constructions, replacing $\mathcal{F}$ by $\mathcal{F}_H$, the filter of H-stable ideals of R with zero annihilator. Thus one obtains Q_H^ℓ, Q_H^r, and finally Q_H, the H-symmetric ring of quotients of R. Note that $Q_H \subset Q$, and similarly $Q_H^\ell \subset Q^\ell$ and $Q_H^r \subset Q^r$.

3.3 Lemma: *Let φ be a twisted action of H on R.*

(1) *Assume φ is invertible. If θ is $\mathcal{F}$-continuous (resp. $\mathcal{F}_H$-continuous), then φ extends uniquely to a twisted action on Q^r (resp. Q_H^r) as follows: For*

$I \in \mathcal{F}$, g: $I_R \to R$, and $h \in H$, choose $J \in \mathcal{F}$ such that $\theta_{h_{(i)}} J \subseteq I$, for all $h_{(i)}$ in $\Delta_3 h$. Then define $h \cdot g$: $J \to R$ by

$$(h \cdot g)a = \sum_h h_{(1)} \cdot [g(\theta_{h_{(2)}} a)] \in R,$$

for all $a \in J$.

(2) *Assume φ is anti-invertible. If ψ is $\mathcal{F}$-continuous (resp. $\mathcal{F}_H$-continuous), then φ extends uniquely to a twisted action on Q^ℓ (resp. Q^ℓ_H) as follows: For $I \in \mathcal{F}$, f: ${}_R I \to R$, and $h \in H$, choose $J \in \mathcal{F}$ such that $\psi_{h_{(i)}} J \subseteq I$, for all $h_{(i)}$ in $\Delta_3 h$. Then define $h \cdot f$: $J \to R$ by*

$$a(h \cdot f) = \sum_h h_{(2)} \cdot [(\psi_{h_{(1)}} a) f] \in R,$$

for all $a \in J$.

Proof: (1) The proof follows the outline of Cohen's argument [C1, Theorem 18] for the case when R is an H-module algebra, replacing Sh by θ_h and using the $\mathcal{F}$-continuous hypothesis to replace I by J where necessary. Thus one sees that $h \cdot g \in Q^r$ and this action extends the H-action on R. However, we give the details to see that H measures Q^r since the argument in [C1] contains a small error and the domain of a is not given. Thus let $g, g' \in Q^r$; we may find $I \in \mathcal{F}$ such that g, g': $I \to R$ and thus gg': $I^2 \to R$. Now choose $J \in \mathcal{F}$ such that $(h_i \cdot g')J \subseteq R$ and $\theta_{h_i} J \subseteq I$, and then choose $K \in \mathcal{F}$ such that $\theta_{h_i} K \subseteq J^2$, for all $h_{(i)}$ in $\Delta_4 h$. Then for any $a \in K$,

$$\begin{aligned}\sum_h (h_1 \cdot g)(h_2 \cdot g')a &= \sum (h_1 \cdot g) h_2 \cdot [g' \theta_{h_3} a] \\ &= \sum h_1 \cdot [g \theta_{h_2}(h_3 \cdot (g' \theta_{h_4} a))] \\ &= \sum h_1 \cdot [gg'(\theta_{h_2} a)] \\ &= (h \cdot gg')(a)\end{aligned}$$

(all terms make sense by our choice of J, K and the fact that $h_3 \cdot (g' \theta_{h_4} a) = (h_3 \cdot g')\theta_{h_4} a$, since $h_3 \cdot g' \in Q^r$). Thus $h \cdot gg' = \sum_h (h_1 \cdot g)(h_2 \cdot g')$ as the functions agree on K.

Next we show that the twisted module condition holds in Q^r. Let $g \in Q^r$ and $I \in \mathcal{F}$ with $gI \subseteq R$. Then choose $J \in \mathcal{F}$ such that $\theta_{h_i k_i} J \subseteq I$ for all $h_i k_i$ in

$\Delta_5 hk$. Then for all $a \in J$, $h, k \in H$,

$$\begin{aligned}\sum_h \sigma(h_1,k_1)(h_2k_2\cdot g)a &= \sum \sigma(h_1,k_1)h_2k_2\cdot[g(\theta_{h_3k_3}a)]\\ &= \sum (h_1\cdot(k_1\cdot[g(\theta_{h_3k_3}a)]))\sigma(h_2,k_2) \quad \text{by (1.3)}\\ &= \sum_h h_1\cdot(k_1\cdot[g(\theta_{h_4k_4}a)\theta_{k_2}\theta_{h_2}\sigma(h_3,k_3)]) \quad \text{by 1.1 (1)}\\ &= \sum_h h_1\cdot(k_1\cdot[g(\theta_{h_2k_2}a)(\sigma(h_3,k_3)a)]) \quad \text{by 1.13 (1)}\\ &= \sum_h (h_1\cdot(k_1\cdot g))(\sigma(h_2,k_2)a.\end{aligned}$$

Since this holds for all $a \in J$, it follows that

$$\sum_h \sigma(h_1,k_1)(h_2k_2\cdot g) = \sum \sigma(h_1k_1\cdot g)(h_2,k_2)$$

and thus the TMC holds in Q^r.

Finally, we show that this extension of φ to Q is unique. Thus assume φ' is another extension of φ to Q^r, and write $\varphi'_h g = h * g$, for $g\colon I_R \to R, I \in \mathcal{F}$. Choose J as before and let $a \in J$. Then

$$\begin{aligned}(h*g)a &= \sum (h_1 * g)[h_2\cdot(\theta_{h_3}a)]\\ &= \sum h_1 * [g(\theta_{h_2}a)]\\ &= \sum h_1\cdot[g(\theta_{h_2}a)]\\ &= (h\cdot g)a.\end{aligned}$$

Since $h*g = h\cdot g$ on J, in fact $h*g = h\cdot g$.

(2) To see that $h\cdot f \in Q^\ell$ and that this extends the H-action on R is similar to the argument in [C], on the other side, using ψ. [Ch, 3.3] shows that H measures Q^ℓ when ψ is fully H-stabilizing and his argument works here, with the minor adjustment as above of choosing K so $\psi_{h_{(i)}}K \subseteq IJ$, all $h_{(i)}$ in $\Delta_3 h$. We check the twisted module condition: Let $f \in Q^\ell$ with $I \in \mathcal{F}$ such that $If \subseteq R$, and choose

$J \in \mathcal{F}$ such that $\psi_{h_i k_i} J \subseteq I$ for all $h_i k_i \in \Delta_3 hk$. Then for all $b \in J$, $h, k \in H$,

$$\begin{aligned} b\sum_h (h_1k_1 \cdot f)\sigma^{-1}(h_2,k_2) &= \sum h_2k_2 \cdot [(\psi_{h_1k_1}b)\, f]\, \sigma^{-1}(h_3,k_3) \\ &= \sum \sigma^{-1}(h_2,k_2)\, h_3 \cdot (k_3 \cdot [(\psi_{h_1k_1}b)\, f]) \text{ by (1.3)} \\ &= \sum h_4 \cdot k_4 \cdot \left[(\psi_{k_3}\psi_{h_3}\sigma^{-1}(h_2,k_2))\,(\psi_{h_1k_1}b)\, f\right] \text{ by 1.1(2)} \\ &= \sum h_3 \cdot k_3 \cdot \left[(\psi_{k_2}\psi_{h_2}\left(b\sigma^{-1}(h_1,k_1)\, f\right))\right] \text{ by 1.13(2)} \\ &= \sum b\sigma^{-1}(h_1,k_1)\ \ (h_2 \cdot k_2 \cdot f)\,. \end{aligned}$$

Since this holds for all $b \in J$, it follows that the TMC holds Q^ℓ.

The uniqueness of the extension of φ to Q^ℓ is proved similarly to the argument for Q^r, only using ψ.

Clearly one could replace $\mathcal{F}$-continuity by $\mathcal{F}_H$-continuity throughout. Thus 3.3 is proved. ∎

3.4 THEOREM: *Let φ be a biinvertible twisted action of H on R.*

(1) *If θ and ψ are $\mathcal{F}_H$-continuous, then φ extends uniquely to a twisted action of H on Q_H.*

(2) *If θ and ψ are $\mathcal{F}$-continuous, then φ extends uniquely to a twisted action of H on Q.*

(3) *If φ is $\mathcal{F}$-continuous, then the extension of φ to Q is also $\mathcal{F}$-continuous.*

Proof: We prove (2), as the argument for (1) is the same. We need to define an action of H on Q. To do this, we use 3.3 and the characterization of Q in 3.2. Thus choose $q \in Q$; q is the equivalence class of (f, g), where for some $I \in \mathcal{F}, f\colon {}_RI \to R$, $g\colon I_R \to R$, and $(af)b = a(gb)$, all $a, b \in I$. Consider f and g as representatives of elements in Q^ℓ and Q^r, and use 3.3. Thus we may define $h \cdot q =$ class of $(h \cdot f, h \cdot g)$. To see that $h \cdot q \in Q$, choose $J \in \mathcal{F}$ such that $\psi_{h_i} J \subseteq I$,

$\theta_{h_i} J \subseteq I$ for all h_i in $\Delta_3 h$, and choose $a, b \in J$. Then

$$\begin{aligned}
[a(h \cdot f)]b &= \sum (h_2 \cdot [(\psi_{h_1} a)] f])b \\
&= \sum_h (h_2 \cdot [(\psi_{h_1} a) f]) h_3 \cdot (\theta h_4 b) \\
&= \sum h_2 \cdot [((\psi_{h_1} a) f)(\theta_{h_3} b)] \\
&= \sum h_2 \cdot [(\psi_{h_1} a)(g(\theta_{h_3} b))] \qquad \text{since } \psi_{h_1} a, \theta_{h_3} b \in I \\
&= \sum [h_2 \cdot (\psi_{h_1} a)] \; [h_3 \cdot (g(\theta_{h_4} b))] \\
&= \sum a[h_1 \cdot (g(\theta_{h_2} b))] \\
&= a[(h \cdot g)b].
\end{aligned}$$

By using $I \bigcap J \in \mathcal{F}$ as the ideal, this shows $h \cdot q \in Q$ by 3.2.

The fact that this action measures Q, extends the action on R, and satisfies the twisted module condition now follows from 3.3, as does uniqueness.

(3) Choose an ideal $I \in \mathcal{F}(Q)$, and let $I_1 = I \cap R$. Then it is easy to see that $I_1 \in \mathcal{F}$. For if $rI_1 = 0$, choose any $q \in I$. Then there exists $J \in \mathcal{F}$ such that $qJ \subset R$. Thus $qJ \subseteq I_1$ and so $rqJ = 0$. Since $J \in \mathcal{F}$, it follows that $rq = 0$. But then $rI = 0$, which implies $r = 0$. Similarly $I_1 r = 0$ implies $r = 0$, and thus $I_1 \in \mathcal{F}$.

Now, assume we are given $I \in \mathcal{F}(Q)$ and $h \in H$. Since φ is $\mathcal{F}$-continuous, there exists $J \in F$ such that $h_{(2)} \cdot J \subseteq I \cap R = I_1$ for all $h_{(2)}$ in $\Delta_2 h$. Let $J' = QJQ$; $J' \in \mathcal{F}(Q)$ since J has no annihilator in Q. Now $h \cdot QJQ = \sum (h_{(1)} \cdot Q) \; (h_{(2)} \cdot J) \; (h_{(3)} \cdot Q) \subseteq Q(I \cap R)Q \subseteq I$. Thus φ is $\mathcal{F}(Q)$-continuous. ∎

3.5 COROLLARY: *Let φ be a twisted action of H on R.*

(1) *If H_0 is cocommutative, then φ is fully biinvertible, and the H-action extends uniquely to a biinvertible twisted action on Q_H. The extensions of θ and ψ are $\mathcal{F}_H(Q_H)$-continuous.*

(2) *If H is pointed, then φ is biinvertible, φ, θ and ψ are $\mathcal{F}$-continuous, and the H-action extends uniquely to a biinvertible twisted action on Q. The extensions of φ, θ, and ψ are all $\mathcal{F}(Q)$-continuous.*

Proof: 2.6 gives that φ is fully biinvertible and the appropriate continuity properties on R. 3.4 gives the extension of φ to Q and Q_H. But now apply 2.6 to Q and Q_H to see φ, θ and ψ are $\mathcal{F}(Q)$ or $\mathcal{F}_H(Q_H)$-continuous. ∎

The Corollary raises the question as to whether it is true in general that a biinvertible twisted action (with θ and ψ continuous) extends to a biinvertible action on Q or Q_H. The next lemma shows that θ and ψ extend to one-sided quotient rings.

3.6 LEMMA: *Let φ be a twisted action of H on R.*

(1) *If φ is invertible, then θ extends to Q^ℓ_H; if also φ is $\mathcal{F}$-continuous, then θ extends to Q^ℓ, as follows:*
For $f\colon {}_RI \to R$, $I \in \mathcal{F}$, $h \in H$, choose $J \in \mathcal{F}$ such that $h_{(i)} \cdot J \subseteq I$, all $h_{(i)} \in \Delta_3 h$. Then define $\theta_h f\colon J \to R$ by

$$a(\theta_h f) = \sum_h \theta_{h_1}[(h_2 \cdot a)f] \in R$$

for all $a \in J$.

(2) *If φ is anti-invertible, then ψ extends to Q^r_H; if also φ is $\mathcal{F}$-continuous, then ψ extends to Q^r, as follows:*
For $g\colon I_R \to R, I \in \mathcal{F}, h \in H$, choose $J \in \mathcal{F}$ such that $h_{(i)} \cdot J \subseteq I$, all $h_{(i)}$ in $\Delta_3 h$. Then define $\psi_h g\colon J \to R$ by

$$(\psi_h g)a = \sum_h \psi_{h_2}[g(h_1 \cdot a)]$$

for all $a \in J$.

(3) *If φ is biinvertible and extends to Q^ℓ_H and Q^r_H (respectively, extends to Q^ℓ and Q^r and is $\mathcal{F}$-continuous), and θ and ψ are $\mathcal{F}_H$-continuous (respectively, $\mathcal{F}$-continuous) then θ extends to an inverse of φ on Q^ℓ_H and Q^ℓ and ψ extends to an anti-inverse of φ on Q^r_H and Q^r.*

Proof: (1) First note that $\theta_h f \in Q^\ell$. For, choose $a \in J, r \in R$; then

$$\begin{aligned}(ra)(\theta_h f) &= \sum_h \theta_{h_1}[(h_2 \cdot ra)f] = \sum_h \theta_{h_1}[(h_2 \cdot r)(h_3 \cdot a)f] \\ &= \sum_h \theta_{h_2}(h_3 \cdot r)\theta_{h_1}[(h_4 \cdot a)f] \quad \text{as } \theta \text{ anti-measures } R \\ &= r\sum_h \theta_{h_1}[(h_2 \cdot a)f] = r(a\, \theta_h f).\end{aligned}$$

Thus $\theta_h f$ is a left R-hom of J to R, so determines an element of Q^ℓ or Q^ℓ_H. The proof of (2) is similar.

(3) Now assume that φ is biinvertible. We first check that θ is an inverse for φ. Thus let f, I, and J be as above, and $a \in J$. Then

$$\begin{aligned} a\left(\sum_h \theta_{h_1}\varphi_{h_2} f\right) &= \sum_h a\theta_{h_1}(h_2 \cdot f) = \sum_h \theta_{h_1}[(h_2 \cdot a)(h_3 \cdot f)] \\ &= \sum \theta_{h_1}(h_4 \cdot [\psi_{h_3}(h_2 \cdot a)]f) \quad \text{by 3.3(2)} \\ &= \sum \theta_{h_1}(h_3 \cdot (\varepsilon(h_2)af)) \\ &= \sum \theta_{h_1}(h_2 \cdot (af)) = a\,\varepsilon(h)f \end{aligned}$$

and thus $\theta * \varphi(h) = \varepsilon(h)id$. In the other direction,

$$\begin{aligned} a\left(\sum_h \varphi_{h_1}\theta_{h_2} f\right) &= \sum [h_2 \cdot [(\psi_{h_1} a)(\theta_{h_3} f)] \quad \text{by 3.3(2)} \\ &= \sum [h_2 \cdot (\psi_{h_1} a)][h_3 \cdot (\theta_{h_4} f)] \\ &= \sum \varepsilon(h_1)a\,\varepsilon(h_2)f = a\,\varepsilon(h)f. \end{aligned}$$

Thus θ is an inverse for φ on Q^ℓ.

That ψ is an anti-inverse of φ on Q^r follows similarly. Thus let g, I and J be as above and $b \in J$. Then

$$\begin{aligned} \left[\sum_h \psi_{h_2}(h_1 \cdot g)\right] b &= \sum \psi_{h_3}[(h_1 \cdot g)(h_2 \cdot b)] \\ &= \sum \psi_{h_4} h_1 \cdot [g\theta_{h_2}(h_3 \cdot b)] \quad \text{by 3.3(1)} \\ &= \sum \psi_{h_2} h_1 \cdot (gb) = \varepsilon(h)gb \end{aligned}$$

and

$$\begin{aligned} \left[\sum_h h_2 \cdot (\psi_{h_1} g)\right] b &= \sum h_2 \cdot [(\psi_{h_1} g)(\theta_{h_3} b)] \quad \text{by 3.3(1)} \\ &= \sum (h_2\psi_{h_1} g)(h_3\theta_{h_4} g) = \varepsilon(h)gb. \qquad \blacksquare \end{aligned}$$

It is interesting to note that the extension of θ being an inverse for the extension of φ requires the existence of ψ, and similarly the extension of ψ being an anti-inverse of φ requires θ. However, we do not know in the above situation whether or not θ and ψ stabilize Q or Q_H. The question reduces to the action of the coradical as the next lemma shows.

3.7 LEMMA: *Assume φ is a biinvertible twisted action of H on R and that φ, θ, and ψ are $\mathcal{F}$-continuous (resp. θ and ψ are $\mathcal{F}_H$-continuous).*

(1) *If $\theta_h Q_H \subseteq Q$ (resp. $\theta_h Q_H \subseteq Q_H$) for all $h \in H_0$, then $\theta_h Q \subseteq Q$ (resp. $\theta_h Q_H \subseteq Q_H$) for all $h \in H$.*

(2) *If $\psi_h Q \subseteq Q$ (resp. $\psi_h Q_H \subseteq Q_H$) for all $h \in H_0$, then $\psi_h Q \subseteq Q$ (resp. $\psi_h Q_H \subseteq Q_H$) for all $h \in H$.*

Proof: (1) We use the construction of θ as in 2.4 and 2.5. Thus let $f = \varphi$ and define g' by $g' = \theta$ on H_0 and $g' = 0$ on B, where $H = H_0 \oplus B$ as vector spaces. Then certainly $g'_h Q \subseteq Q$, for all $h \in H$. Also note $h \cdot Q \subseteq Q$ for all h by Theorem 3.4.

Write $\Delta h = \sum_i h_i \otimes h'_i$. Then for any $q \in Q$,

$$(f * g')_h(q) = \sum_i f_{h_i}\left(g'_{h'_i} q\right) \subseteq \sum_i h_i \cdot Q \subseteq Q.$$

Thus for $\gamma = \eta \circ \varepsilon - f * g'$, $\gamma_h Q \subseteq Q$, all h. Since $\theta = g' * \sum_{n \leq 0} \gamma^n$, it follows that $\theta_h Q \subseteq Q$, all $h \in H$. Clearly we could replace Q by Q_H.

(2) Use the fact, as in 2.5, that ψ is the inverse of φ^{op} in $\mathrm{Hom}(H^{op}, \mathrm{End}\, R)$, constructed as above. ∎

When the H-action extends to both quotient rings (as in the case of H pointed; see 3.5) then something can be said about the invariants in Q and Q_H.

3.8 LEMMA: *Let φ be a biinvertible twisted action of H on R, which extends to Q_H and to Q. Then $Q^H = (Q_H)^H$; that is, the invariant rings are the same.*

Proof: Since $Q_H \subset Q, (Q_H)^H \subset Q^H$ trivially. Thus, choose $q \in Q^H$. Since $q \in Q$, there exists $I \in \mathcal{F}$ such that $qI, Iq \subseteq R$. By 1.12, $H \cdot I$ is an H-stable ideal of R containing I. Thus $H \cdot I \in \mathcal{F}_H$. Thus it will suffice to show $q(H \cdot I), (H \cdot I)q \subseteq R$. Since $q \in Q(R)^H$,

$$h \cdot (qa) = \sum_h (h_1 \cdot q)(h_2 \cdot a) = \sum_h \varepsilon(h_1) q(h_2 \cdot a) = q(h \cdot a),$$
$$\text{all } h \in H, \quad a \in R.$$

Thus $q(h \cdot I) = h \cdot (qI) \subseteq h \cdot R \subseteq R$. Similarly $(H \cdot I)q \subseteq R$ and so $q \in Q_H$. ∎

4. Crossed product actions

In this section we specialize to a crossed product action φ; recall σ must satisfy the cocycle condition (1.4). We first consider some sufficient conditions for φ to be biinvertible, generalizing a result of [Ch]. We then consider crossed products over the various quotient rings of R.

Thus, let $R\#_\sigma H$ be a crossed product, as in (1.5). Let $\gamma \in \mathrm{Hom}(H, R\#_\sigma H)$ denote the map $h \mapsto 1\#h$. It is proved in [BM] that γ is (convolution) invertible whenever σ is invertible, though we shall not need the actual formula for γ^{-1}. It is also noted in [BM], [BCM] that for all $h \in H$, $a \in R$,

$$\varphi_h a = h \cdot a = \sum_h \gamma(h_1) a \gamma^{-1}(h_2). \tag{4.1}$$

We will see that θ and ψ can take similar forms. The first part of the next theorem is due to Chin [Ch, 1.3].

4.2 THEOREM: *Let $R\#_\sigma H$ be a crossed product with action φ and let γ be as above.*

(1) *If γ has an anti-inverse $\hat{\gamma}$, then φ has an anti-inverse given by*

$$\psi_h a = \sum_h \hat{\gamma}(h_2) a \gamma(h_1).$$

(2) *If γ^{-1} has an anti-inverse $\tilde{\gamma}$, then φ has an inverse given by*

$$\theta_h a = \sum_h \gamma^{-1}(h_2) a \tilde{\gamma}(h_1).$$

Moreover, ψ_h and θ_h stabilize all H-stable ideals of R. Thus φ is fully biinvertible whenever γ and $\gamma - 1$ are anti-invertible.

Before proving the theorem, we consider a basic example of $\hat{\gamma}$ and $\tilde{\gamma}$.

4.3 Example: Assume that the cocycle σ is trivial. Then γ^{-1} is always anti-invertible, with $\tilde{\gamma}(h) = 1\#S^2 h$. For, recall that in this case $\gamma^{-1}(h) = 1\#Sh$. Then

$$\sum_h \gamma^{-1}(h_2)\tilde{\gamma}(h_1) = \sum 1\#Sh_2 S^2 h_1 = \sum 1\#S((Sh_1)h_2)$$
$$= 1\#S(\varepsilon(h)\cdot 1) = \varepsilon(h)1\#1.$$

Similarly

$$\sum_h \tilde{\gamma}(h_2)\gamma^{-1}(h_1) = \varepsilon(h)1\#1.$$

If also S is bijective, then γ is always anti-invertible with $\hat{\gamma}(h) = 1\#S^{-1}h$; this follows from the fact that S^{-1} is an anti-inverse for id_H in $\mathrm{Hom}(H, H)$.

Returning to the theorem, note that it is straightforward to check that ψ and θ as defined in 4.2 are a formal inverse and anti-inverse for φ; the difficulty lies in showing that $\psi_h a$ and $\theta_h a$ are elements of R, for all $a \in R$, $h \in H$. Chin proves this for $\psi_h a$ by using [DT, Prop 5]. For $\theta_h a$ we must first prove an extension of Doi and Takeuchi's result.

First we need some notation. As before, for C a coalgebra and A an algebra, the convolution product in $\mathrm{Hom}(C, A)$ is denoted by $*$. The anti-convolution product will be denoted by $\times$; that is for $f, g \in \mathrm{Hom}(C, A)$,

$$(f \times g)(c) = \sum_c f(c_2)g(c_1).$$

In this notation, the above definitions mean that

$$\gamma \times \hat{\gamma} = \hat{\gamma} \times \gamma = \varepsilon \cdot id \quad \text{and} \quad \gamma^{-1} \times \tilde{\gamma} = \tilde{\gamma} \times \gamma^{-1} = \varepsilon \cdot id.$$

Now consider a right H-comodule algebra A, with structure map ρ: $A \to A \otimes H$. As in [DT], we also consider the two algebra maps i_1: $A \to A \otimes H$ given by $i_1(a) = a \otimes 1$ and i_2: $H \to A \otimes H$ given by $i_2(h) = 1 \otimes h$. Observe that i_2 has a convolution inverse $i_2^{-1}(h) = 1 \otimes Sh$, and as in Example 4.3 i_2^{-1} has an anti-inverse $\tilde{i}_2(h) = 1 \otimes S^2 h$. We apply these maps when $A = R\#_\sigma H$, with $\rho(r\#h) = \sum_h r\#h_1 \otimes h_2$.

4.4 Lemma: *Consider $A = R\#_\sigma H$, with ρ, γ, i_1 and i_2 as above. Then*

(1) [DT] $\rho \circ \gamma^{-1} = i_2^{-1} * (i_1 \circ \gamma^{-1})$.

(2) *if $\tilde{\gamma}$ exists, then* $\rho \circ \tilde{\gamma} = (i_1 \circ \tilde{\gamma}) * \tilde{i}_2$.

Proof: (1) is [DT, Prop. 5]. For (2), since ρ is an algebra map and $\tilde{\gamma}$ is the anti-inverse of γ^{-1}, it follows that $\rho \circ \tilde{\gamma}$ is the anti-inverse of $\rho \circ \gamma^{-1}$. Now (2) will follow from (1), provided we show that $(i_1 \circ \tilde{\gamma}) * \tilde{i}_2$ is the anti-inverse of $i_2^{-1} * (i_1 \circ \gamma^{-1})$. To see this, note that the image of i_1 in $A \otimes H$ commutes with the images of i_2^{-1} and of $\tilde{i}_2$. Then for any $h \in H$,

$$\begin{aligned}
\left[(i_1 \circ \tilde{\gamma}) * \tilde{i}_2\right] \times \left[i_2^{-1} * \left(i_1 \circ \gamma^{-1}\right)\right](h) &= \sum_h i_1\tilde{\gamma}(h_3)\tilde{i}_2(h_4)i_2^{-1}(h_1)i_1\gamma^{-1}(h_2) \\
&= \sum_h i_1\tilde{\gamma}(h_3)i_1\gamma^{-1}(h_2)\tilde{i}_2(h_4)i_2^{-1}(h_1) \\
&= \sum \left(\tilde{\gamma}(h_3)\gamma^{-1}(h_2)\otimes 1\right)\tilde{i}_2(h_4)i_2^{-1}(h_1) \\
&= \sum \varepsilon(h_2)(1\#1\otimes 1)\tilde{i}_2(h_3)i_2^{-1}(h_1) \\
&= \varepsilon(h)(1\#1\otimes 1).
\end{aligned}$$

By uniqueness of anti-inverses, (2) is proved. ∎

Proof of Theorem 4.2: As discussed above, we must show $\theta_h r \in R$ for all $r \in R, h \in H$. Considering $A = R\#_\sigma H$ as an H-comodule algebra as above, we know that $R = R\#1 = \{a \in A|\ \rho(a) = a\otimes 1\}$, the coinvariants of A [BCM, 5.10]. Thus it suffices to show $\rho(\theta_h r) = \theta_h r \otimes 1$. Now

$$\begin{aligned}
\rho(\theta_h r) &= \rho\left(\sum_h \gamma^{-1}(h_2)r\tilde{\gamma}(h_1)\right) \\
&= \sum_h \rho\circ\gamma^{-1}(h_2)\rho(r)\rho\circ\tilde{\gamma}(h) \\
&= \sum_h \left[i_2^{-1} * \left(i_1\circ\gamma^{-1}\right)\right](h_2)\rho(r)\left[(i_1\circ\tilde{\gamma}) * \tilde{i}_2\right](h_1) \\
&= \sum_h i_2^{-1}(h_3)i_1\gamma^{-1}(h_4)i_1(r\#1)i_1\tilde{\gamma}(h_1)\tilde{i}_2(h_2) \\
&= \sum_h i_1\left(\gamma^{-1}(h_4)(r\#1)\tilde{\gamma}(h_1)\right)i_2^{-1}(h_3)\tilde{i}_2(h_2) \\
&= \sum_h i_1\left(\gamma^{-1}(h_3)r\tilde{\gamma}(h_1)\right)\varepsilon(h_2) \\
&= i_1(\theta_h r) = \theta_h r\otimes 1.
\end{aligned}$$

Thus $\theta_h r \in R$ and so $\theta \in \mathrm{Hom}(H, \mathrm{End}\, R)$.

Finally if I is an H-stable ideal of R, then $\theta_h I \subseteq R \cap AIA \subseteq R\cap(I\#_\sigma H) = I$. Thus I is θ_h-stable. ∎

The theorem raises the question as to when γ and γ^{-1} are anti-invertible; in fact this is not always the case, see [MSc]. We do have the following:

4.5 LEMMA: *γ and γ^{-1} are anti-invertible provided their restrictions to H_0 are anti-invertible in* $\mathrm{Hom}(H_0, \mathrm{End}\, R)$. *In particular this happens if H_0 is cocommutative.*

Proof: We use Lemma 2.4 applied to $\mathrm{Hom}(H_0^{cop}, \mathrm{End}\, R)$ as in 2.5 to prove the first statement. Now if H_0 is cocommutative, then $\hat{\gamma} = \gamma^{-1}$ and $\tilde{\gamma} = \gamma$ on H_0, proving the lemma. ∎

We note that Chin [Ch. 1.4] proved γ was anti-invertible when $H_0 = k \cdot 1$ (that is, H is irreducible).

For the rest of this section, we consider a crossed product action φ which is fully biinvertible; by 3.3 and 3.4, φ extends to a twisted action on Q_H^r, Q_H^ℓ, and Q_H. Since the cocycle condition extends trivially, in fact these extensions of φ are all crossed product actions; thus one may form the crossed products $Q_H^r\#_\sigma H, Q_H^\ell\#_\sigma H$ and $Q_H\#_\sigma H$. We wish to study the relationship between these crossed products and the ordinary right, left, and symmetric quotient rings of $R\#_\sigma H$. First we need a lemma: part (2) is in [Ch].

4.6 LEMMA: *Let φ be an anti-invertible crossed product action of H on R. Then*

(1) *$R\#_\sigma H \cong H \otimes R$ as right R-modules,*

(2) *$I\#H = (1\#H)(I\#1)$ for any H-stable, ψ_H-stable ideal I of R,*

(3) *if $I \in \mathcal{F}_H$, then $I\#H \in \mathcal{F}(R\#_\sigma H)$.*

Proof: (1) The argument is the same as [KT, 1.6], replacing S^{-1} by the anti-inverse ψ of φ. That is, define α: $H \otimes R \to R\#_\sigma H$ by $\alpha(h \otimes r) = (1\#h)\ (r\#1)$ and $\alpha^{-1} = \beta$ by $\beta(r\#h) = \Sigma h_2 \otimes \psi_{h_1} r$. It is easy to check that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity, and clearly α is a right R-map, where $R\#_\sigma H$ is a right R-module via right multiplication by $r\#1$, any $r \in R$.

(2) is [Ch. 1.5] and is similar to showing $\alpha \circ \beta = id$ in (1). That is, for $a \in I, h \in H$,

$$a\#h = \sum_h h_2 \cdot (\psi_{h_1} a)\#h_3 = \sum_h (1\#h_2)\ (\psi_{h_1} a\#1) \in (1\#H)\ (I\#1)$$

and $(I\#H)\ (I\#1) \subseteq I\#H$ since I is H-stable.

(3) Choose $w \in R\#_\sigma H$ such that $(I\#1)w = 0$. If $w = \sum a_i\#h_i$ then $0 = (I\#1)w = \sum_i Ia_i\#h_i$. Since $I \in \mathcal{F}_H$, $Ia_i = 0$ implies $a_i = 0$, all i, and thus $w = 0$. If $w(I\#H) = 0$, then $w(I\#1) = 0$. By part (1), the isomorphism β: $R\#_\sigma H \to$

$H \otimes R$ takes $r\#1$ to $1 \otimes r$. Thus $0 = w(I\#1) = \beta(w(I\#1)) = \beta(w)(1 \otimes I)$. But $\beta(w) = \sum h_i \otimes r_i$, some $h_i \in H$, $r_i \in R$, and so $0 = \sum_i h_i \otimes r_i I$. Then $r_i I = 0$, all i implies $r_i = 0$ and so $w = 0$ as before. Thus $I\#H \in \mathcal{F}(R\#_\sigma H)$. ∎

4.7 THEOREM: *Let φ be a fully biinvertible crossed product action of H on R. Then each crossed product above embeds into the corresponding quotient of $R\#\sigma H$; that is,*

(1) $Q_H^r(R)\#_\sigma H \hookrightarrow Q^r(R\#_\sigma H)$,
(2) $Q_H^\ell(R)\#_\sigma H \hookrightarrow Q^\ell(R\#_\sigma H)$,
(3) $Q_H(R)\#_\sigma H \hookrightarrow Q(R\#_\sigma H)$.

Proof: We prove (3), since the others are similar. We first choose $w \in Q_H(R)$ and show that it can be extended to an element $\hat{w} \in Q(R\#_\sigma H)$. Since $w \in Q_H(R)$, there exists $I \in \mathcal{F}_H$ such that $wI, Iw \subseteq R$. Let $K = I\#H$; then $K \in \mathcal{F}(R\#_\sigma H)$ by 4.6. Now define $\hat{w}: K \to R\#_\sigma H$ by $\hat{w}(a\#h) = w(a)\#h$ for all $a \in I, h \in H$. We claim that $\hat{w}$ is a right $R\#H$-map. For choose $r \in R$, $k \in H$. Then

$$\begin{aligned}\hat{w}\left((a\#h)(r\#k)\right) &= \hat{w}\left(\sum_{h,k} a\,(h_1 \cdot r)\,\sigma(h_2,k_1)\,\#h_3k_2\right)\\ &= \sum_{h,k} w\left(a\,(h_1 \cdot r)\,\sigma(h_2,k_1)\right)\#h_3k_2\\ &= w(a)\sum_{h}(h_1 \cdot r)\,\sigma(h_2,k_1)\,\#h_3k_2\\ &= (w(a)\#h)(r\#k) = (\hat{w}(a\#h))(r\#k)\end{aligned}$$

where we have used that w is a right R-map. Thus $\hat{w} \in Q(R\#_\sigma H)$.

To see $\hat{w} \in Q(R\#_\sigma H)$, it suffices to show that $L\hat{w} \subseteq R\#_\sigma H$, for some $L \in \mathcal{F}(R\#_\sigma H)$. First, we note that if $a \in I$, then $(a\#1)\hat{w} = aw\#1$. For if $b\#h \in K$, then

$$(a\#1)\hat{w}(b\#h) = (a\#1)\;(wb\#h) = awb\#h = (aw\#1)(b\#h).$$

Thus $[(a\#1)\hat{w} - (aw\#1)]K = 0$ in $Q^r(R\#_\sigma H)$; it follows that $(a\#1)\hat{w} = aw\#1$. Let $u = (1\#h)(a\#1)$, for $h \in H$, $a \in I$. For any $b\#k \in I\#H = K$,

$$\begin{aligned}u\hat{w}\,(b\#k) &= (1\#h)\,(a\#1)\,\hat{w}\,(b\#k) = (1\#h)\,(a\#1)\,(wb\#k)\\ &= (1\#h)\,(awb\#k) = (1\#h)\,(a\#1)\,(b\#k).\end{aligned}$$

Thus $[u\hat{w} - (1\#h)(aw\#1)]K = 0$, and so $u\hat{w} = (1\#h)(aw\#1) \in R\#H$. Thus $K\hat{w} \subseteq R\#H$ and so $\hat{w} \in Q(R\#_\sigma H)$.

Finally we claim that the map π: $Q_H(R)\#_\sigma H \to Q(R\#_\sigma H)$ given by $\pi(\sum_i w_i\#h_i) = \sum_i \hat{w}_i(1\#h_i)$ is a monomorphism, where the $\{h_i\}$ are linearly independent. For, choose $I \in \mathcal{F}$ such that $Iw_i \subseteq R$, all i. Then if $\sum w_i\#h_i \in$ Kerπ and $a \in I$,

$$0 = \sum_i \hat{w}_i(1\#h_i) = \sum_i (a\#1)\hat{w}_i(1\#h_i) = \sum aw_i\#h_i \in R\#_\sigma H.$$

Thus $aw_i = 0$, all i, all a. Since $I \in \mathcal{F}_H$, this gives $w_i = 0$, all i, and so Ker $\pi = 0$. It is straightforward to check that π is a homomorphism. ∎

Some sufficient conditions for $Q_H(R)\#_\sigma H = Q(R\#_\sigma H)$ are given in [BeM].

Now consider the centers of these quotient rings. $C(R) = Z(Q(R))$ is called the **extended center** of R, and $C_H(R) = Z(Q_H(R))$ is the H-extended center. In fact $C_H(R) = C(R) \cap Q_H(R)$, since $R \subset Q_H(R) \subset Q(R)$. Moreover C (respectively C_H) is also the center of Q^ℓ and Q^r (resp. of Q^ℓ_H and Q^r_H). Although in general the center C_H of Q_H is not H-stable, we use the notation $C_H(R)^H$ to mean $C_H(R) \cap Q_H(R)^H$; that is the central H-invariants of $Q_H(R)$.

4.8 COROLLARY: *Under the embeddings described in 4.7,*

$$C_H(R)^H\#1 \hookrightarrow C(R\#_\sigma H).$$

Proof: Choose $\lambda \in C_H(R)^H$; by the proof of 4.7, $\hat{\lambda} \in Q(R\#_\sigma H)$. We claim $\hat{\lambda} \in C(R\#_\sigma H)$. Since in general C is the centralizer of R in Q, it suffices to show that $\hat{\lambda}$ centralizes $R\#_\sigma H$. Since $\lambda \in C_H(R)$, it centralizes $R = R\#1$. For $1\#H$, choose $h \in H$. Identifying $\hat{\lambda}$ with $\lambda\#1$,

$$(1\#h)(\lambda\#1) = \sum_h h_1 \cdot \lambda\#h_2 = \sum \varepsilon(h_1)\lambda\#h_2 = \lambda\#h = (\lambda\#1)(h\#1),$$

and thus $\hat{\lambda}$ centralizes $1\#H$. ∎

We close by comparing our quotient actions and crossed products with Chin's. He proceeds as follows: Beginning with a fully anti-invertible crossed product action φ, he extends φ to Q^ℓ_H, and then constructs a left quotient $Q^\sim$ of $R\#_\sigma H$ using the filter $\mathcal{F}^\sim = \{I\#H \mid I \in \mathcal{F}_H\}$. Then $Q^\ell_H\#_\sigma H$ embeds in $Q^\sim$, so must be associative, and thus is a crossed product. Using this crossed product, he then shows that the H-action on Q_H^ℓ restricts to one on Q_H by using the map $\gamma\colon H \to R\#_\sigma H$ as in 4.1. Our methods are more direct, in that we first extend twisted actions to Q_H and Q_H^ℓ, then construct $Q_H\#_\sigma H$ and $Q_H^\ell\#_\sigma H$, and finally embed these crossed products in $Q(R\#_\sigma H)$ and $Q^\ell(R\#_\sigma H)$. It is not clear that Chin's $Q^\sim$ is the same as $Q^\ell(R\#_\sigma H)$ since the filter $\mathcal{F}^\sim$ may be smaller than $\mathcal{F}(R\#_\sigma H)$.

In fact it is now known [MSc] that any crossed product action extends to the quotient rings Q_H^r, Q_H^ℓ, and Q_H provided S is bijective; in that case biinvertibility is not necessary. Nevertheless it would still be of interest to know when γ and γ^{-1} are anti-invertible, since in that case many formulas have a simpler form.

References

[A] S. A. Amitsur, *Rings of quotients*, Symposia Mathematica **VIII** (1972), 149–164.

[BeM] J. Bergen and S. Montgomery, *Ideals and quotients in crossed products of Hopf algebras*, J. Algebra **151** (1992), 374–439.

[BCM] R. J. Blattner, M. Cohen and S. Montgomery, *Crossed products and inner actions of Hopf algebras*, Trans. Am. Math. Soc. **298** (1986), 671–711.

[BM] R. J. Blattner and S. Montgomery, *Crossed products and Galois extensions of Hopf algebras*, Pacific J. Math. **137** (1989), 37–54.

[Ch] W. Chin, *Crossed products and generalized inner actions of Hopf algebras*, Pacific J. Math. **150** (1991), 241–259.

[C] M. Cohen, *Hopf algebras acting on semiprime algebras*, AMS Contemporary Math. **43** (1985), 49–61.

[C2] M. Cohen, *Smash products, inner actions, and quotient rings*, Pacific J. Math. **125** (1986), 45–65.

[DT] Y. Doi and M. Takeuchi, *Cleft comodule algebras for a bialgebra*, Comm. Algebra **14** (1986), 801–817.

[KT] H. F. Kreimer and M. Takeuchi, *Hopf algebras and Galois extensions of an algebra*, Indiana Math. J. **30** (1981), 675–692.

[M] S. Montgomery, *Fixed rings of finite automorphism groups of associative rings*, Lecture Notes in Math. Vol. 818, Springer-Verlag, Berlin, 1980.

[M2] S. Montgomery, *Some remarks on filtrations of Hopf algebras*, Comm. Algebra, to appear.

[MSc] S. Montgomery and H.-J. Schneider, *manuscript*, in preparation.

[P] D. S. Passman, *Infinite Crossed Products,* Academic Press, Boston, 1989.

[R] D. E. Radford, *Irreducible representations of $Uq(g)$ arising from $Mod_{C^{\frac{1}{2}}}$*, to appear in Proceedings of the Conference on Quantum Deformations of Algebras and their Representations, held at Bar-Ilan University, January 1992.

[Sc] H.-J. Schneider, *On inner actions of Hopf algebras and stabilizers of representations,* J. Algerba, to appear.

[S] M. E. Sweedler, *Hopf Algebras,* Benjamin, New York, 1969.

[T] M. Takeuchi, *Free Hopf algebras generated by coalgebras,* J. Math. Soc. Japan **23** (1971), 561–582.

[TW] E. J. Taft and R.E. Wilson, *On antipodes in pointed Hopf algebras,* J. Algebra **29** (1974), 27–32.